

Persistent current in a 2D Josephson junction array wrapped around a cylinder

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We study persistent currents in a Josephson junction array wrapped around a cylinder. The $T = 0$ quantum statistical mechanics of the array is equivalent to the statistical mechanics of a classical xy spin system in 2+1 dimensions at the effective temperature $T^* = \sqrt{2JU}$, with J being the Josephson energy of the junction and U being the charging energy of the superconducting island. It is investigated analytically and numerically on lattices containing over one million sites. For weak disorder and $T^* \ll J$ the dependence of the persistent current on disorder and T^* computed numerically agrees quantitatively with the analytical result derived within the spin-wave approximation. The high- T^* and/or strong-disorder behavior is dominated by instantons corresponding to the vortex loops in 2+1 dimensions. The current becomes destroyed completely at the quantum phase transition into the Cooper-pair insulating phase.

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I. INTRODUCTION

Persistent currents in closed chains of Josephson junctions (JJ) have been studied for more than two decades (see, e.g. Refs. 1–5 and references therein). Being conceptually similar to small superconducting rings they provide a testing ground for models of quantum phase slips [6–9] and superconductor-insulator transition (SIT) [10–12]. This research has significantly intensified in recent years due to advances in manufacturing of nanostructures [13] and the renewed interest to quantum phase transitions [14, 15] inspired in part by the prospects of applications of quantum circuitry [16–18].

There exist even a greater volume of work on SIT in disordered ultrathin films (see, e.g. Ref. 19 and references therein). Similar to superconducting rings, various mechanisms of SIT have been modeled by two-dimensional JJ arrays [20–24]. To have a persistent current in a 2D JJ array the latter should be closed into a cylindrical surface that encloses the magnetic flux, see Fig. 1. In this Letter we argue that studies of persistent currents in such a system provides another avenue for testing the theory of quantum phase transitions. It may also be relevant to properties of an ultrathin superconducting film deposited on a cylinder and properties of a superconducting topological insulator [25].

A system that bears some relevance to the JJ array wrapped around a cylinder is a JJ ladder made of capacitively coupled one-dimensional JJ rings, e.g., JJ necklaces stuck together. It was studied theoretically by the numerical density-matrix renormalization group with an emphasis on the role of excitons [26]. In this Letter we are taking a different approach. When the dynamics of the 2D JJ array is dominated by capacitances of superconducting islands its $T = 0$ quantum statistical mechanics is equivalent [1, 10, 14, 27, 28] to the statistical mechanics of the classical xy spin system in 2 + 1 dimensions at

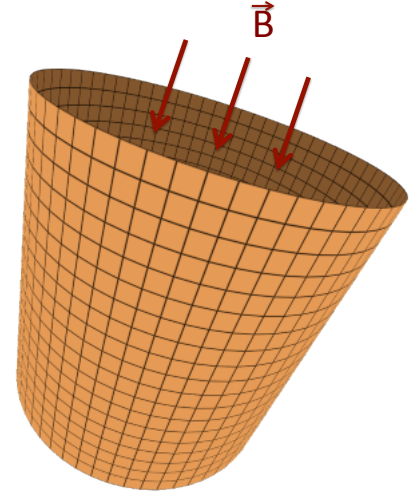


Figure 1: Color online: A 2D JJ array wrapped around a cylinder that encloses the magnetic flux.

the effective temperature $T^* = \sqrt{2JU}$, with J being the Josephson energy of the junction and U being the charging energy of the superconducting island. The advantage of such a mapping is its suitability for large-scale Monte Carlo (MC) studies.

To make this problem relevant to experimental systems it must also include disorder. Positional disorder in a planar JJ array is known to give rise to random phases when the array is placed in the transverse magnetic field [29]. The resulting “gauge” or “Bose” glass has been intensively studied by analytical [30–33] and numerical [28, 34, 35] methods. Phase disorder has been found to have a significant effect on the SIT. The model with the transverse magnetic field would not apply to the JJ array wrapped around a cylinder. However, static random

phases in Josephson junctions can also be generated by other mechanisms, e.g., by the broken time-reversal symmetry in the presence of the magnetic moments [36, 37].

At first we neglect disorder. Let θ_{ij} be the phase of the superconducting order parameter $\Psi = |\Psi| \exp(i\theta)$ at the ij -th superconducting island, with $i = 1, 2, \dots, N$ denoting the islands in the j -th ring ($j = 1, 2, \dots, N$) of the 2D JJ array shown in Fig. 1. The Josephson energy of the array is [38]

$$E_J = J \sum_{ij} [1 - \cos(\theta_{i,j+1} - \theta_{ij})] + J \sum_{ij} \left[1 - \cos \left(\theta_{i+1,j} - \theta_{ij} + \frac{2\pi}{\Phi_0} \int_i^{i+1} \mathbf{A} \cdot d\mathbf{l}_j \right) \right], \quad (1)$$

where the vector potential \mathbf{A} is due to the magnetic flux Φ enclosed by the cylinder.

Summation along each j -th ring gives

$$\sum_i (\theta_{i+1,j} - \theta_{ij}) = 2\pi m_j, \quad \frac{2\pi}{\Phi_0} \sum_i \int_i^{i+1} \mathbf{A} \cdot d\mathbf{l}_j = 2\pi\phi, \quad (2)$$

where $\phi \equiv \Phi/\Phi_0$, $\Phi_0 = h/(2e)$ is the flux quantum, and m_j is an integer. This allows one to write

$$E_J = J \sum_{ij} [1 - \cos(\tilde{\theta}_{i,j+1} - \tilde{\theta}_{ij})] + J \sum_{ij} \left[1 - \cos \left(\tilde{\theta}_{i+1,j} - \tilde{\theta}_{ij} + \frac{2\pi(\phi + m_j)}{N} \right) \right], \quad (3)$$

where the reduced phases $\tilde{\theta}$ are defined in such a way that the change of $\tilde{\theta}_i$ around the j -th ring is zero. The total accumulation of the original phase θ_{ij} in a closed path around the cylinder is accounted for by the quantum number m_j . The persistent current in the cylinder is given by

$$I = \frac{d\langle E_J \rangle}{d\Phi} = \frac{1}{\Phi_0} \frac{d\langle E_J \rangle}{d\phi}, \quad (4)$$

The energy minimum corresponds to all $\tilde{\theta}_{ij}$ and all m_j being the same ($m_j = m$), leading to the ground state

$$E_J^{(0)} = JN^2 \left[1 - \cos \left(\frac{2\pi(\phi + m)}{N} \right) \right] \cong 2\pi^2 J(\phi + m)^2, \quad (5)$$

the last expression being the large- N case. Branches of $E_J^{(0)}(\Phi)$ and $I^{(0)}(\Phi) = dE_J^{(0)}/d\Phi$ for different values of m are shown in Fig. 2a and Fig. 2b respectively. When $\phi = n + 1/2$, with n being an integer, that is for $\Phi = (n + 1/2)\Phi_0$, the ground state is degenerate, $E_J(n, m) = E_J(n, m' = -2n - m - 1)$. For, e.g., half a fluxon, $\Phi =$

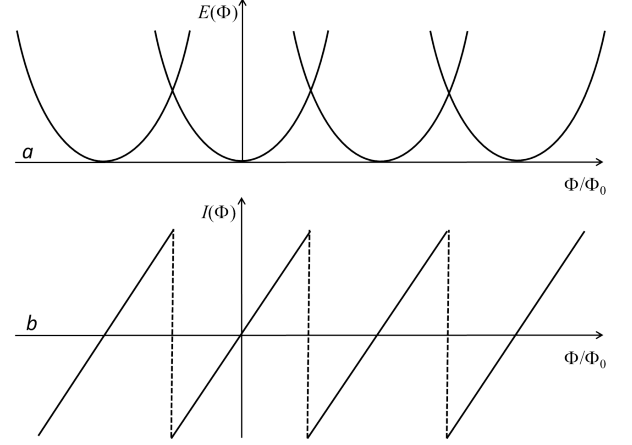


Figure 2: (a) m -branches of the ground-state energy $E_J^{(0)}$. (b) m -branches of the persistent current.

$\Phi_0/2$, the current has the same absolute value but flows in opposite directions for $m = 0$ and $m = -1$.

The dynamics of the JJ array is due to the electrical charging of the superconducting islands by the excess (or lack) of Cooper pairs n_{ij} at the ij -th site. It is determined by the finite capacitances of the islands to the ground and the capacitances of the junctions. Different limits, with both capacitances present, can be achieved in experiment and have been studied in literature, see, e.g., Ref. 4 and references therein. In this paper we are considering the limit in which the capacitances of the islands, C , greatly exceed the capacitances of the junctions. In this case the charging energy is given by

$$E_C = \sum_{ij} U n_{ij}^2 = \frac{\hbar^2}{4U} \sum_{ij} \left(\frac{d\theta_{ij}}{dt} \right)^2 \quad (6)$$

where $U = (2e)^2/(2C)$. The second of Eq. (6), which plays the role of the kinetic energy, is obtained by noticing that n_{ij} and θ_{ij} are canonically conjugated variables, $n_{ij} = -i d/d\theta_{ij}$, so that $i\hbar d\theta_{ij}/dt = [\theta_{ij}, E_C] = 2iU n_{ij}$. Finite U permits quantum tunneling between the current states that correspond to $E_J(n, m)$ and $E_J(n, m')$.

The Lagrangian of the model is $\mathcal{L} = E_C - E_J$. Its quantum mechanics is formulated in terms of the path integral

$$\mathcal{I} = \prod_i \int D\{\theta_i(\tau)\} e^{-S_E/\hbar} \quad (7)$$

where $\tau = it$ and $S_E = \int d\tau \mathcal{L}$ is the Euclidean action with

$$\mathcal{L} = \frac{\hbar^2}{4U} \sum_{ij} \left(\frac{d\theta_{ij}}{d\tau} \right)^2 + J \sum_{ij} [1 - \cos(\theta_{i,j+1} - \theta_{ij})] + J \sum_{ij} \left[1 - \cos \left(\theta_{i+1,j} - \theta_{ij} + \frac{2\pi\phi}{N} \right) \right]. \quad (8)$$

Statistical mechanics of this quantum model at $T = 0$ is equivalent [1, 10, 14, 27, 28] to the statistical mechanics of the classical model in 2+1 dimensions at a non-zero temperature $T^* = \sqrt{2JU}$, described by the partition function

$$Z = \prod_i \int D\{\theta_i(\tau)\} e^{-\mathcal{H}_{2+1}/T^*} \quad (9)$$

with $\mathcal{H}_{1+1} = -\frac{1}{2} \sum_{\mathbf{r}\mathbf{r}'} J_{\mathbf{r}\mathbf{r}'} \cos(\theta_{\mathbf{r}'} - \theta_{\mathbf{r}} + \phi_{\mathbf{r}\mathbf{r}'})$, where \mathbf{r} is a discrete three-dimensional vector $\mathbf{r} = (i, j, l)$ representing the space-time lattice, while $J_{\mathbf{r}\mathbf{r}'} = J$ for the nearest neighbors and zero otherwise. In the numerical work we use the $N \times N \times N$ lattice with the l direction corresponding to the imaginary time and periodic boundary conditions. Non-zero phase shifts are given by $\phi_{i,j,l;i\pm 1,j,l} = \pm 2\pi\phi/N$. Notice that, in principle, the periodic boundary condition imposed on the imaginary time introduces a finite physical temperature into the original quantum problem, $T \sim T^*/N$. At large N the effect of that temperature on the persistent current can be ignored.

The statistical model presented above can be reformulated in terms of the three-component classical spin vectors of the 3D xy model at temperature $T^* = \sqrt{2JU}$ that describes the strength of quantum fluctuations. That model has a ferromagnetic-paramagnetic phase transition at $T^* = T_c$ that in the original model corresponds to the quantum phase transition into the Cooper-pair insulator state in which the islands connected by Josephson junctions maintain their superconductivity but no Josephson current can circulate around the cylinder. The natural way to test this prediction is to study the dependence of the persistent current on U .

Phase slips corresponding to quantum tunneling between different m require formation of vortex loops in 2+1 dimensions. At small U satisfying $T^* = \sqrt{2JU} \ll T_c \sim J$ such loops nucleate with exponentially small probability. Consequently, at a small T^* , if one induces a persistent current by placing the cylinder with the JJ array in the magnetic field, the phase slips may not occur on the time scale of the experiment. In this case all m_j in Eq. 3 are the same, $m_j = m$, and the persistent current computed with the help of Eq. (4) and the symmetry becomes

$$I = \frac{2\pi NJ}{\Phi_0} \sin \left[\frac{2\pi(\phi + m)}{N} \right] \langle \cos(\tilde{\theta}_{i,j,l} - \tilde{\theta}_{i+1,j,l}) \rangle. \quad (10)$$

We now recall that the statistical mechanics of our model is that of the 3D xy model at $T = T^*$, for which the low-temperature (spin-wave) result for the cubic lattice is $\langle \cos(\theta_{\mathbf{r}} - \theta_{\mathbf{r}+\delta}) \rangle = 1 - T^*/(6J)$, with δ being the nearest neighbor in any direction. This gives for large N

$$I \cong \frac{(2\pi)^2 J}{\Phi_0} (\phi + m) \left(1 - \frac{T^*}{6J} \right). \quad (11)$$

We shall now consider the effect of quenched disorder by adding a static random phase ϕ_{ij} to the phase difference, $\theta_{i,j+1} - \theta_{ij}$, between neighboring superconducting islands i and j in Eq. (1). In the models of planar JJ arrays such random phase can be generated by the positional disorder in the 2D lattice of Josephson junctions in the presence of the transverse magnetic field [28, 29, 31–35]. In our case of a JJ array wrapped around a cylinder, random phases can be generated due to, e.g., anomalous Josephson effect in the presence of magnetic moments [36, 37].

As was shown in Ref. 30 the continuous counterpart of the model with quenched randomness corresponds to the interaction of the continuous phase order parameter $\theta(\mathbf{r})$ with a static random field $\mathbf{q}(\mathbf{r}) = \phi_{ij}\hat{r}_{ij}/a$ (a being the lattice spacing), described by

$$E_J = \frac{1}{2} J \int d^2r [\nabla\theta(\mathbf{r}) - \mathbf{q}(\mathbf{r})]^2. \quad (12)$$

The Imry-Ma argument [39] favors the destruction of the long-range order in less than four dimensions by a weak static random field interacting with the order parameter directly, since at $d < 4$ such interaction, regardless of strength, dominates the energy at large distances. Crucial to that argument, however, is the formation of topological defects which makes the order more robust [40]. In our case, random field in Eq. (12) interacts with the gradient of the order parameter, which further diminishes its effect at large distances. One should expect, therefore, that the initially ordered state will not be destroyed by weak quenched randomness at low T^* .

The persistent current in Eq. (10) should now be calculated with account of averaging over phase fluctuations generated by the random field. Introducing random phases ϕ_{ij} into Eq. (3), similarly to the above in the limit of large N we obtain

$$I = \frac{(2\pi)^2 J (\phi + m)}{\Phi_0} \left\langle \cos \left[\tilde{\theta}_{i+1,j,l} - \tilde{\theta}_{i,j,l} + \phi_{ij} \right] \right\rangle. \quad (13)$$

In the case $|\phi_{ij}| \ll 1$ and $|\tilde{\theta}_{i+1,j,l} - \tilde{\theta}_{i,j,l}| \ll 1$ the continuous model of Eq. (12) yields

$$I = \frac{(2\pi)^2 J (\phi + m)}{\Phi_0} \left\{ 1 - \frac{1}{2} a^2 \langle [\nabla\theta(\mathbf{r}) - \mathbf{q}(\mathbf{r})]^2 \rangle \right\}. \quad (14)$$

The effects of low temperature and weak static randomness must be additive. It suffices, therefore, to compute the contribution of quenched randomness at $T^* = 0$ just minimizing the energy, Eq. (12). The extrema of the latter satisfy the equation

$$\nabla^2\theta(\mathbf{r}) = \nabla \cdot \mathbf{q}(\mathbf{r}) \quad (15)$$

having the solution

$$\theta(\mathbf{r}) = \int d^2r' G(\mathbf{r}-\mathbf{r}') \nabla \cdot \mathbf{q}(\mathbf{r}') = \int d^2r' \mathbf{q}(\mathbf{r}') \cdot \nabla G(\mathbf{r}-\mathbf{r}'), \quad (16)$$

where $G(\mathbf{r}) = (2\pi)^{-1} \ln(r/a)$ is the Green function of the 2D Laplace equation, $\nabla^2 G(\mathbf{r}) = \delta(\mathbf{r})$. With the help of this equation one obtains

$$\nabla_\alpha \theta(\mathbf{r}) - q_\alpha(\mathbf{r}) = \int d^2 r' q_\beta(\mathbf{r}') [\nabla_\alpha \nabla_\beta G(\mathbf{r} - \mathbf{r}') - \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}')] , \quad (17)$$

so that

$$\begin{aligned} [\nabla \theta(\mathbf{r}) - \mathbf{q}(\mathbf{r})]^2 &= \int d^2 r' \int d^2 r'' q_\beta(\mathbf{r}') q_\gamma(\mathbf{r}'') \\ &\times [\nabla_\alpha \nabla_\beta G(\mathbf{r} - \mathbf{r}') - \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}')] \\ &\times [\nabla_\alpha \nabla_\gamma G(\mathbf{r} - \mathbf{r}'') - \delta_{\alpha\gamma} \delta(\mathbf{r} - \mathbf{r}'')] . \end{aligned} \quad (18)$$

We shall assume that

$$\langle q_\beta(\mathbf{r}') q_\gamma(\mathbf{r}'') \rangle = \frac{1}{2} a^2 q_R^2 \delta_{\beta\gamma} \delta(\mathbf{r}' - \mathbf{r}'') . \quad (19)$$

Then

$$\begin{aligned} \langle [\nabla \theta(\mathbf{r}) - \mathbf{q}(\mathbf{r})]^2 \rangle &= \frac{1}{2} a^2 q_R^2 \int d^2 r [\nabla_\alpha \nabla_\beta G(\mathbf{r}) - \delta_{\alpha\beta} \delta(\mathbf{r})]^2 \\ &= \frac{1}{2} a^2 q_R^2 \int \frac{d^2 k}{(2\pi)^2} [k_\alpha k_\beta G(\mathbf{k}) + \delta_{\alpha\beta} \delta(\mathbf{k})]^2 \\ &= \frac{1}{2} a^2 q_R^2 \int \frac{d^2 k}{(2\pi)^2} \left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right]^2 \\ &= \frac{1}{2} a^2 q_R^2 \int \frac{d^2 k}{(2\pi)^2} = \frac{1}{2} q_R^2 , \end{aligned} \quad (20)$$

where we have used Fourier transforms: $G(\mathbf{k}) = -1/k^2$ and $\delta(\mathbf{k}) = 1$. Thus

$$I = \frac{(2\pi)^2 J (\phi + m)}{\Phi_0} \left(1 - \frac{1}{4} a^2 q_R^2 \right) . \quad (21)$$

The last step is to establish the relation between q_R and ϕ_R used in the numerical experiment. If ϕ_{ij} are chosen randomly between $-\phi_R$ and ϕ_R , one has

$$a^2 q_R^2 = \frac{1}{2\phi_R} \int_{-\phi_R}^{+\phi_R} d\phi \phi^2 = \frac{\phi_R^2}{3} . \quad (22)$$

Inserting this into the above formula and combining it with Eq. (11), one finally obtains the persistent current at low T^* in the presence of weak randomness

$$I = \frac{(2\pi)^2 J (\phi + m)}{\Phi_0} \left(1 - \frac{T^*}{6J} - \frac{\phi_R^2}{12} \right) . \quad (23)$$

As we shall see, this formula agrees well with numerical results.

Higher T^* and ϕ_R tend to create topological defects. Destruction of the persistent current by the latter and spin waves can be investigated numerically using the

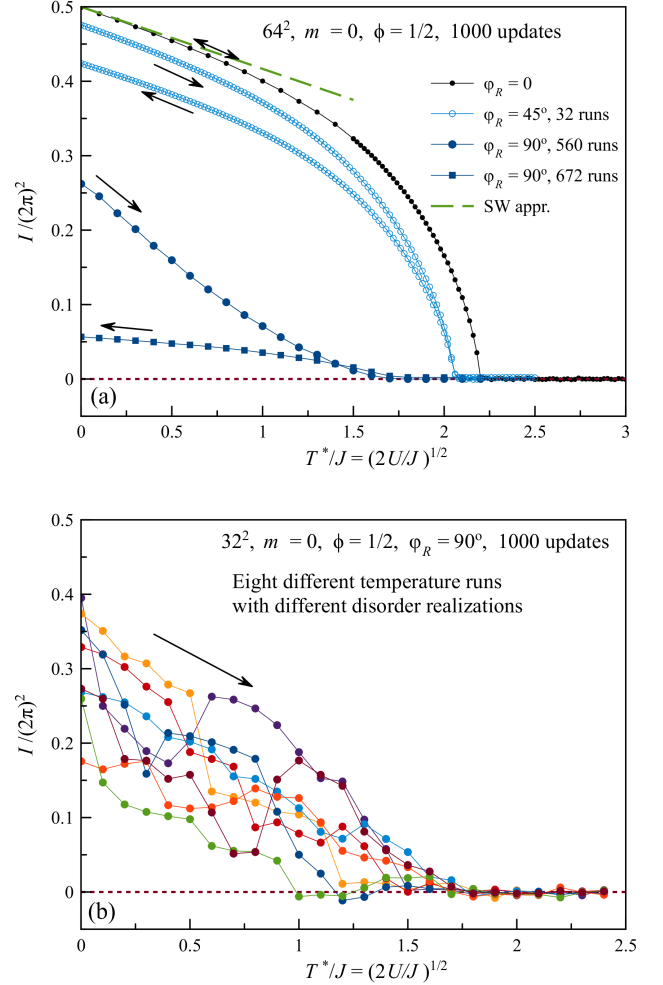


Figure 3: Color online: Effective-temperature dependence of the persistent current. (a) Averaged over disorder realizations (runs). (b) Individual runs showing strong fluctuations for strong disorder.

equivalent magnetic model of two-component vectors \mathbf{s} of fixed length instead of the grain phase θ .

The case $T^* = 0$ is classical, so that one can minimize the energy of the 2D system of Josephson grains by the method of Ref. 5 that uses successive rotations of the grain's vectors \mathbf{s} into the direction of the effective field \mathbf{H}_{eff} with probability α and overrelaxation (reflecting \mathbf{s} with respect to \mathbf{H}_{eff}) with probability $1 - \alpha$. Small α provide the highest efficiency. In this work we used $\alpha = 0.01$ everywhere. The results of energy minimization are averaged over realizations of the disorder φ_{ij} . For this, many runs were done until averages stabilized and smooth curves, such as $I(T^*)$, were obtained.

For non-zero T^* the original 2D quantum problem was solved as the effective classical 3D problem with the effective temperature T^* using the standard Monte Carlo procedure. For each classical spin (Josephson grain) we

used the Monte Carlo update with probability $\alpha = 0.01$ and overrelaxation with probability $1 - \alpha$. For each parameter value (T^* or φ_R) at least 1000 system updates were performed. This was sufficient for local equilibration and averaging over thermal fluctuations. On top of it, averaging over realizations of disorder (runs) was performed.

In the case of strong disorder the system demonstrates glassy properties, so that relaxation leads to different final states depending on the initial state and other factors. Here Monte Carlo routine does not lead to global equilibration for $T^* \lesssim J$. To average out glassy fluctuations, one has to perform many runs with different realizations of disorder. The numerical problem for the persistent current I is tougher than computation of the magnetization of a ferromagnet. With a given number of fluxons ϕ , the current is inversely proportional to the length of the rings N and directly proportional to the number of the rings N , so that I is independent of N , Eq. (23). Thus increasing the system size does not lead to strong suppression of fluctuations, as in the case of the magnetization. Performing a large number of repeated measurements (runs) is the only way to beat fluctuations.

Fig. 3a shows dependence of the persistent current I on the effective temperature T^* , obtained by increasing or decreasing T^* in small steps starting from the collinear spin state (same phase everywhere, $m = 0$). In the absence of disorder the process of temperature change is reversible and Eq. (11) is a good approximation at $T^*/J \ll 1$. The current I vanishes at $T^*/J = 2.22$ that corresponds to the ferromagnetic transition in 3D xy model. In contrast to the magnetization, no finite-size effects are seen, the curves being practically the same for $N = 32, 64, 128$. For $\phi_R = 45^\circ$ there is a hysteresis and the transition point is moving down, $T^*/J = 2.06$. For $\phi_R = 90^\circ$ hysteresis is very strong and the transition point is difficult to detect. Fig. 3b reflects strong fluctuations from run to run for such strong disorder.

Fig. 4 shows the dependence of the persistent current on the disorder strength φ_R at $T^* = 0$, computed for the 2D classical model, and for $T^*/J = 1$. All results are obtained starting from the collinear initial condition (CIC) for each φ_R value. The two curves cross because of glassy effects and different computational methods used. At $T^* = 0$ minimization of the energy of the 2D system occurs faster and better than Monte Carlo in 3D, leading to slightly lower energies. Perfect match of the curves for different sizes shows the absence of size effects. Eq. (23) works well for small disorder, actually up to $\varphi_R \simeq 60^\circ$. In this region vortex loops begin to pop up and vorticity f_V (see, e.g., Ref. [5]) starts to grow from zero. At $\varphi_R \simeq 100^\circ$ the persistent current practically vanishes.

In Conclusion, we have proposed a novel system for the study of quantum phase transitions: Josephson junction

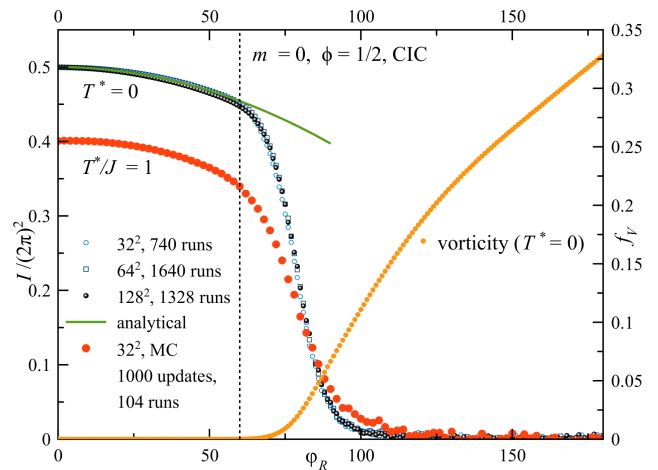


Figure 4: Color online: Disorder dependence of the persistent current and vorticity at different effective temperatures.

array wrapped around a cylinder. The dependence of the persistent current on the Josephson and charging energies and on the strength of disorder, computed analytically in the spin wave approximation, agrees well with numerical results. Cases of strong quantum fluctuations and strong disorder have been studied numerically on lattices having over one million sites. Quantum phase transition in the Josephson junction array wrapped around a cylinder is dominated by instantons corresponding to the vortex loops in 2+1 dimensions. Experimental study of such a system would be of great interest.

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- [1] M. Y. Choi, Persistent current and voltage in a ring of Josephson junctions, *Physical Review B* **48**, 15920-15925 (1993).
 - [2] K. A. Matveev, A. I. Larkin, and L. I. Glazman, Persistent current in superconducting nanorings, *Physical Review Letters* **89**, 096802-(4) (2002).
 - [3] I. M. Pop, I. Protopopov, F. Lecocq, Z. Peng, B. Pannetier, O. Buisson, and W. Guichard, Measurement of the effect of quantum phase-slips in a Josephson junction chain, *Nature Physics* **6**, 589-592 (2010).
 - [4] G. Rastelli, I. M. Pop, and F. W. J. Hekking, Quantum phase-slips in Josephson junction rings, *Physical Review B* **87**, 174513-(18) (2013).
 - [5] D. A. Garanin and E. M. Chudnovsky, Quantum decay of the persistent current in a Josephson junction ring, *Physical Review B* **93**, 094506-(9) (2016).
 - [6] A. D. Zaikin, D. S. Golubev, A. van Otterlo, and G. T. Zimanyi, Quantum phase slips and transport in ultrathin superconducting wires, *Physical Review Letters* **78**, 1552-1555 (1997).

- [7] D. S. Golubev and A. D. Zaikin, Quantum tunneling of the order parameter in superconducting nanowires, *Physical Review B* **64**, 014504-(14) (2001).
- [8] G. Schwiete and Y. Oreg, Persistent current in small superconducting rings, *Physical Review Letters* **103**, 037001-(4) (2009), and references therein.
- [9] A. G. Semenov and A. D. Zaikin, Persistent currents in quantum phase slip rings, *Physical Review B* **88**, 054505-(10) (2013).
- [10] R. M. Bradley and S. Doniach, Quantum fluctuations in chains of Josephson junctions, *Physical Review B* **30**, 1138-1147 (1984).
- [11] S. E. Korshunov, Effect of dissipation on the low-temperature properties of a tunnel-junction chain, *Soviet Physics JETP* **68**, 609-618 (1989).
- [12] E. Chow, P. Delsing, and D. B. Haviland, Length-scale dependence of the superconductor-to-insulator quantum phase transition in one dimension. *Physical Review Letters* **81**, 204-207 (1998).
- [13] See review and references therein: J. E. Mooij, G. Schöen, A. Shnirman, T. Fuse, C. J. P. M. Harmans, H. Rotzinger, and A. H. Verbruggen, Superconductor-insulator transition in nanowires and nanowire arrays, *New Journal of Physics* **17**, 033006-(12) (2015).
- [14] S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, Continuous quantum phase transitions, *Review of Modern Physics* **69**, 315-333 (1997).
- [15] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, UK, 2011).
- [16] L. B. Ioffe, M. V. Feigel'man, A. Ioselevich, D. Ivanov, M. Troyer, and G. Blatter, Topologically protected quantum bits using Josephson junction arrays, *Nature* **415**, 503-506 (2002).
- [17] S. Gladchenko, D. Olaya, E. Dupont-Ferrier, B. Douçot, L. B. Ioffe, and M. E. Gershenson, Superconducting nanocircuits for topologically protected qubits, *Nature Physics* **5**, 48-53 (2009).
- [18] V. E. Manucharyan, J. Koch, L. I. Glazman, and M. H. Devoret, Fluxonium: single Cooper-pair circuit free of charge offsets, *Science* **326**, 113-116 (2009).
- [19] Y.-H. Lin, J. Nelson, and A. M. Goldman, Superconductivity of very thin films: The superconductor-insulator transition, *Physica C* **514**, 130-141 (2015).
- [20] R. Fazio and G. Schön, *Physical Review B* **43**, 5307 (1991).
- [21] Y. Imry, M. Strongin, and C. C. Homes, An inhomogeneous Josephson phase in thin-film and high-Tc superconductors, *Physica C* **468**, 288-293 (2008).
- [22] M. V. Fistul, V. M. Vinokur, and T. I. Baturina, *Physical Review Letters* **100**, 086805 (2008).
- [23] S. V. Syzranov, K. B. Efetov, and B. L. Altshuler, dc conductivity of an array of Josephson junctions in the insulating state, *Physical Review Letters* **103**, 127001-(4) (2009).
- [24] E. M. Chudnovsky, Instanton glass generated by noise in a Josephson junction array, *Physical Review Letters* **103**, 137001-(4) (2009).
- [25] B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors*, Princeton University Press (Princeton, NJ, 2013).
- [26] M. Lee, M.-S. Choi, and M. Y. Choi, Quantum phase transitions and persistent currents in Josephson-junction ladders, *Physical Review B* **68**, 144506-(11) (2003).
- [27] M. Wallin, E. S. Sørensen, S. M. Girvin, and A. P. Young, Superconductor-insulator transition in two-dimensional dirty boson systems, *Physical Review B* **49**, 12115-12139 (1994).
- [28] K. Kim and D. Stroud, Quantum Monte Carlo study of a magnetic-field-driven two-dimensional superconductor-insulator transition, *Physical Review* **78**, 174517-(16) (2008).
- [29] E. Granato and J. M. Kosterlitz, Quenched disorder in Josephson-junction arrays in a transverse magnetic field, *Physical Review B* **33**, 6533-6536 (R) (1986).
- [30] M. Rubinstein, B. Shraiman, and D. R. Nelson, Two-dimensional XY-magnets with random Dzyaloshinskii-Moriya interactions, *Physical Review B* **27**, 1800-1811 (1983).
- [31] S. E. Korshunov and T. Nattermann, Phase diagram of a Josephson junction array with positional disorder, *Physica B* **222**, 280-286 (1996).
- [32] M. S. Li, T. Nattermann, H. Rieger, and M. Schwartz, Vortex lines in the three-dimensional XY model with random phase shifts, *Physical Review B* **54**, 16026-16031 (1996).
- [33] P. Gupta and S. Teitel, Positional disorder in the fully frustrated Josephson junction array: Random Gaussian Phase Shifts in the Fully Frustrated 2D XY Model, *Physical Review Letters* **82**, 5313-5316 (1999).
- [34] L.-H. Tang and P. Tong, Zero-temperature criticality in the two-dimensional gauge glass model, *Physical Review Letters* **94**, 207204-(4) (2005).
- [35] K. Kim and D. Stroud, Continuous phase transition of a fully frustrated XY model in three dimensions, *Physical Review* **73**, 224504-(10) (2006).
- [36] A. Buzdin, Direct coupling between magnetism and superconducting current in the Josephson φ_0 junction, *Physical Review Letters* **101**, 107005-(4) (2008).
- [37] F. Konschelle and A. Buzdin, Magnetic moment manipulation by a Josephson current, *Physical Review Letters* **102**, 017001 (2009).
- [38] M. Tinkham, *Introduction to Superconductivity* (Dover Publications, 2004, ISBN 0-486-43503-2).
- [39] Y. Imry and S.-k. Ma, Random-field instability of the ordered state of continuous symmetry, *Physical Review Letters* **35**, 1399-1401 (1975).
- [40] T. C. Proctor, D. A. Garanin, and E. M. Chudnovsky, Random fields, topology, and the Imry-Ma argument, *Physical Review Letters* **112**, 097201-(4) (2014).